Discrete Applied Mathematics 157 (2009) 1509–1520

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Gray codes for non-crossing partitions and dissections of a convex polygon

Clemens Huemer [a,*], Ferran Hurtado [a], Marc Noy [a], Elsa Omaña-Pulido [b]

[a] Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya (UPC), Jordi Girona,1-3, 08034 Barcelona, Spain

[b] Universidad Autónoma Metropolitana, México D.F., Mexico

ARTICLE INFO

Article history:
Received 21 July 2006
Received in revised form 18 May 2008
Accepted 3 June 2008
Available online 26 July 2008

Keywords:
Gray code
Hamiltonian cycle
Non-crossing partition
Polygon dissection

ABSTRACT

In this paper we develop Gray codes for two families of geometric objects: non-crossing partitions and dissections of a convex polygon by means of non-crossing diagonals.

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1. Introduction

In recent years much work has been done on finding Gray codes for many families of combinatorial objects. Many examples can be found in the comprehensive survey [17]; more recent works are [1,2,4,6,11,15,16]. In this paper we obtain new Gray codes for two families of geometric objects: non-crossing partitions and polygon dissections.

A partition of the set $[n] = \{1, 2, \ldots, n\}$ is called *non-crossing* if given four elements $a < b < c < d$ with a and c belonging to a block, then b and d do not both belong to a different block. Let $\mathrm{NC}(n)$ and $\mathrm{NC}(n, k)$ be, respectively, the set of all non-crossing partitions of $[n]$ and those having exactly k blocks. If A and B are two blocks in a non-crossing partition, and $x \in A$, we say that *x is visible from B* (or that B can see x) if the replacement of A and B by $A \setminus \{x\}$ and $B \cup \{x\}$, respectively, results in a non-crossing partition.

We define graphs on the vertex sets $\mathrm{NC}(n)$ and $\mathrm{NC}(n, k)$ by declaring two partitions adjacent if they differ by the move of a single element from one block to another block or by the exchange of visible elements between two blocks, one element from each of them. When we only consider the first operation we have the *constrained graphs* $\mathrm{CNC}(n)$ and $\mathrm{CNC}(n, k)$. Our main result in Section 2 is that $\mathrm{NC}(n)$ and $\mathrm{CNC}(n)$ are Hamiltonian, i.e. contain a Hamilton cycle, for every $n \geq 3$ and that $\mathrm{NC}(n, k)$ is Hamiltonian for every $k \geq 3$ and $n > k$. On the other hand, $\mathrm{NC}(n, 2)$ is Hamiltonian for $n > 3$ while $\mathrm{CNC}(n, 2)$ is not, and we conjecture that $\mathrm{CNC}(n, k)$ is Hamiltonian for $n > k \geq 3$.

For combinatorial set partitions, Gray codes are known in the literature from several sources. Either for the set of all partitions of $[n]$, or for partitions of $[n]$ into k blocks; see [3,10,14,17].

A *dissection* of a convex polygon P_n with n vertices is a subdivision of P_n into convex subpolygons by means of a set of k non-crossing diagonals, where $0 \leq k \leq n - 3$. Let $\mathrm{D}(n)$ and $\mathrm{D}(n, k)$ denote, respectively, the set of all dissections of P_n and

* Corresponding author. Tel.: +34 934137695.
E-mail addresses: clemens.huemer@upc.edu (C. Huemer), ferran.hurtado@upc.edu (F. Hurtado), marc.noy@upc.edu (M. Noy), eop@xanum.uam.mx (E. Omaña-Pulido).

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doi:10.1016/j.dam.2008.06.018

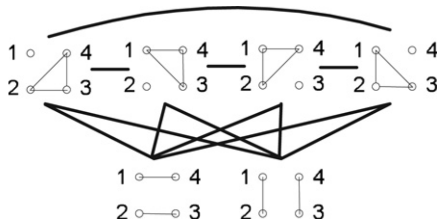

Fig. 1. The graph NC(4, 2).

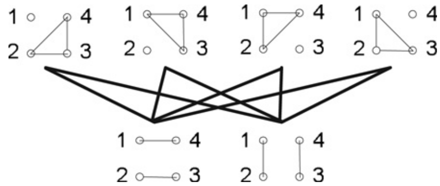

Fig. 2. The graph CNC(4, 2).

those using exactly k diagonals. Define two dissections δ and δ' in $D(n, k)$ adjacent if one can be obtained from the other by means of the following operation: remove a diagonal e from δ and add a new one inside the polygon formed by merging the two subpolygons adjacent to e. We prove in Section 3 that the resulting graph $D(n, k)$ is Hamiltonian for every $k \geq 1$ and $n \geq 4$. If we allow diagonals to be removed from a dissection without replacement, then we have a graph on $D(n)$ and we show that it is also Hamiltonian for every $n \geq 4$.

In particular, for $k = n - 3$ this implies that the *graph of triangulations* of a convex polygon is Hamiltonian, a result first proved in [13] for which a much simpler proof was obtained in [7].

As we mention later, non-crossing partitions can also be viewed as geometric configurations on the vertices of a convex polygon. Thus in a certain way this paper concludes a systematic study of graphs defined on such configurations, see [7–9].

2. Non-crossing partitions

A partition of the set $[n] = \{1, 2, \ldots, n\}$ is a family $\pi = \{B_1, \ldots, B_k\}$ of pairwise disjoint non-empty subsets of $[n]$ such that $[n] = B_1 \cup \cdots \cup B_k$. The B_i are called *blocks* of the partition, and we say that π is a partition into k blocks. A *singleton* is a block of the form $\{a\}$ and a *doubleton* is a block of the form $\{a, b\}$.

A partition is *non-crossing* (NC for short) if $a < b < c < d$ and a block contains a and c, then no other block contains both b and d. Geometrically, if one draws the elements of $[n]$ on the boundary of a circle and a block B is represented as a convex polygon having as vertices the elements of B, then the condition is equivalent to the fact that no two blocks cross. If A and B are two blocks in a non-crossing partition, and $x \in A$, we say that *x is visible from B* (or that B can see x) if the replacement of A and B by $A \setminus \{x\}$ and $B \cup \{x\}$, respectively, results in a non-crossing partition.

We denote by $\mathrm{NC}(n)$ the set of all NC-partitions of $[n]$, and by $\mathrm{NC}(n, k)$ those that have exactly k blocks. There is a vast literature on NC-partitions, since they are related to a variety of topics, ranging from lattices to polytopes and probability theory (see [18] for a thorough recent survey). We just remark here that $|\mathrm{NC}(n)| = \frac{1}{n+1}\binom{2n}{n}$ and that $|\mathrm{NC}(n, k)| = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$, the Catalan and Narayana number, respectively.

We now define two graphs on the set $\mathrm{NC}(n)$. We say that two *NC*-partitions of $[n]$ are adjacent if they differ by taking an element from one block and moving it into another block (possibly creating a singleton) or by the exchange of visible elements between two blocks, one element from each of them. We denote the resulting graph by the same symbol $\mathrm{NC}(n)$. When we consider only the first operation we obtain the *constrained graph* $\mathrm{CNC}(n)$, which is obviously a spanning subgraph of $\mathrm{NC}(n)$. The graphs NC(4, 2) and CNC(4, 2) are shown in Figs. 1 and 2.

Our first result is that these graphs are Hamiltonian. In order to prove it we need some definitions and preliminary lemmas.

Place a new vertex $n + 1$ between n and 1 at its right place. Given a partition $P \in \mathrm{NC}(n)$, the *children* of P are defined as follows: for every block B of P seen by vertex $n + 1$, create a new partition by including $n + 1$ into B. There is yet one more child P^* of P defined by adding the singleton block $\{n + 1\}$ to P. Observe that every partition in $\mathrm{NC}(n + 1)$ has a unique parent in $\mathrm{NC}(n)$, obtained by removing $n + 1$. The set of children of P is denoted $C(P)$.

The following lemmas are immediate from the definitions, and we omit the proofs.

Lemma 1. *The children $C(P)$ of a partition $P \in \mathrm{NC}(n)$ induce a complete subgraph in* $\mathrm{CNC}(n)$.

Lemma 2. *If P and Q are adjacent in* $\mathrm{CNC}(n)$, *then P^* and Q^* are adjacent in* $\mathrm{CNC}(n + 1)$.

Lemma 3. *If P and Q are adjacent in* $\mathrm{CNC}(n)$*, then there exist $\widehat{P} \neq P^*$ child of P and $\widehat{Q} \neq Q^*$ child of Q, such that $\widehat{P}$ and $\widehat{Q}$ are adjacent in* $\mathrm{CNC}(n+1)$.

For $n \geq 3$, define two special partitions P_n and P'_n as follows:

$$P_n = \{\{1, 2, \dots, n\}\}, \qquad P'_n = \{\{1\}, \{2, 3, \dots, n\}\}.$$

Observe that P_{n+1} is a child of P_n and that P'_{n+1} is a child of P'_n.

Theorem 4. *For $n \geq 3$, there exists a Hamilton cycle in* $\mathrm{CNC}(n)$ *where P_n and P'_n are adjacent.*

Proof. The proof is by induction on n; the case $n = 3$ is shown in Fig. 3. Assume the result holds for $n \geq 3$, and let $P_n, P'_n, P, Q, \dots, R, P_n$ be a Hamilton cycle in $\mathrm{CNC}(n)$. If we apply the operation $P \to P^*$ to this cycle we obtain, because of Lemma 2, a cycle $\mathcal{C}$ in $\mathrm{CNC}(n+1)$, that we write as

$$(P_n)^*, (P'_n)^*, P^*, Q^*, \dots, R^*, (P_n)^*.$$

By the previous lemmas and observations, the vertices of $\mathrm{CNC}(n+1)$ are grouped into disjoint subsets $C(P)$, one for each partition $P \in \mathrm{NC}(n)$, and each $C(P)$ has a unique element P^* in $\mathcal{C}$. The construction of the Hamilton cycle depends on the parity of $|\mathrm{NC}(n)|$.

If $|\mathrm{NC}(n)|$ is even, the construction is as follows. Start with P_{n+1} and go to P'_{n+1}. Next traverse the remaining vertices in $C(P'_n)$ in any order ending at $(P'_n)^*$ (this can be done thanks to Lemma 1). Then go to P^*, the other neighbor of $(P'_n)^*$ in $\mathcal{C}$, traverse the remaining vertices in $C(P)$, and jump to a vertex in $C(Q)$ different from Q^* (see Lemma 3). Visit the reaming vertices in $C(Q)$ ending at Q^*, go to the other neighbor of Q^* in $\mathcal{C}$, and repeat the process. The evenness condition guarantees that at the end we can jump from R^* to $(P_n)^*$, traverse the remaining vertices in $C(P_n)$, and finish at P_{n+1}, thus completing a Hamilton cycle.

If $|\mathrm{NC}(n)|$ is odd, we need a small modification. We note that $C(P'_n)$ has only three elements, namely $(P'_n)^*$, P'_{n+1} and $P''_{n+1} = \{\{1, n+1\}, \{2, 3, \dots, n\}\}$.

Again start with P_{n+1} and go to P'_{n+1}, then to $(P'_n)^*$, then to P''_{n+1}. Next go to an element in $C(P)$ different from P^*, then traverse the remaining elements in $C(P)$ ending at P^*. Then go to Q^*, visit the remaining vertices in $C(Q)$, and continue the process. As can be seen, the difference is that now we use the edge connecting P^* and Q^*, and from there every second edge in $\mathcal{C}$. At the end we finish in R^*, jump to $(P_n)^*$, and conclude in P'_{n+1} as before. This case is illustrated in Fig. 3. □

Since $\mathrm{CNC}(n)$ is a spanning subgraph of $\mathrm{NC}(n)$ we immediately obtain the following result.

Corollary 5. *The graph* $\mathrm{NC}(n)$ *is Hamiltonian for every $n \geq 3$.*

Consider $\mathrm{NC}(n, k)$ and $\mathrm{CNC}(n, k)$ as subgraphs of $\mathrm{NC}(n)$ and $\mathrm{CNC}(n)$, respectively. The adjacency rule is the same as in $\mathrm{NC}(n)$ and $\mathrm{CNC}(n)$ but, since the number of blocks has to remain equal to k, one is not allowed to add a singleton to a block or to take an element from one block and create a singleton. In the rest of this section we study the hamiltonicity of these graphs. Observe that $\mathrm{NC}(n, 1)$ and $\mathrm{NC}(n, n)$ are reduced to a single vertex.

Theorem 6. *For every $k \geq 2$ and every $n > k$ the graph* $\mathrm{NC}(n, k)$ *contains a Hamilton cycle.*

2.1. *Proof of* Theorem 6

We use induction to prove Theorem 6. Given Hamilton cycles in $\mathrm{NC}(n, k)$ and $\mathrm{NC}(n, k+1)$ we construct a Hamilton cycle in $\mathrm{NC}(n+1, k+1)$. As basis of the induction we show that the graphs $\mathrm{NC}(n, 2)$ and $\mathrm{NC}(n, n-1)$ contain a Hamilton cycle for all $n \geq 4$. The scheme of the induction is shown in Fig. 4.

2.1.1. *The graph* $\mathbf{NC}(\boldsymbol{n}, 2)$

As notation for any partition we will write only the elements of one (usually the smaller) block; for example, the partition $\{\{1, 2, 3, 4\}, \{5, 6\}\}$ will be denoted as $\{5, 6\}$. The number of elements in the smaller block of a partition is named *block size*. We call the union of all partitions having the same block size *level*. For every $i \in \{1, \dots, n\}$ the singleton $\{i\}$ has degree 4. It is adjacent to $\{i-1, i\}$, $\{i, i+1\}$, $\{i-1\}$ and $\{i+1\}$ (mod n). Every other partition $\{i, \dots, j\}$ has degree 6. It is adjacent to $\{i-1, i, \dots, j\}$, $\{i, \dots, j, j+1\}$, $\{i+1, \dots, j\}$, $\{i, \dots, j-1\}$, $\{i+1, \dots, j+1\}$, $\{i-1, \dots, j-1\}$. We arrange the nodes of the graph in levels. The first level contains all singletons, the second level consists of doubletons, and so on. The graph has $\lfloor n/2 \rfloor$ levels. If n is odd then every level consists of n partitions. If n is even then all but the last level contain n partitions and the last level has $n/2$ partitions. The graph $\mathrm{NC}(6, 2)$ is depicted in Fig. 5.

Lemma 7. *There exists a Hamilton cycle in* $\mathrm{NC}(n, 2)$ *for all $n \geq 4$ in which no exchange of the element 1 is done.*

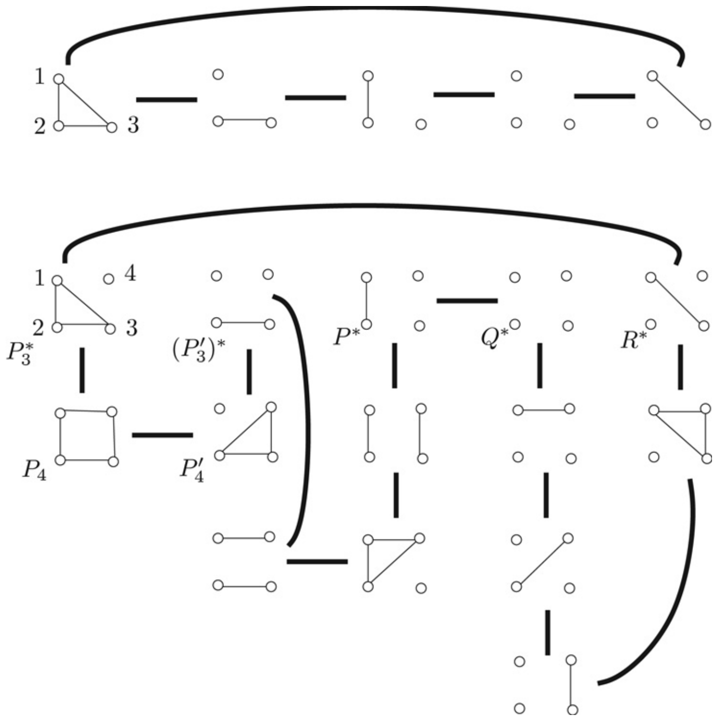

Fig. 3. A Hamilton cycle in CNC(3) and in CNC(4).

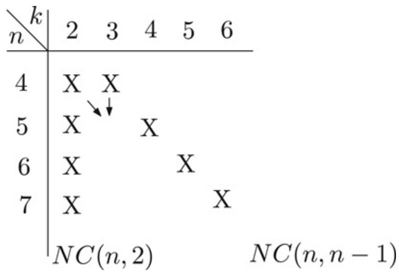

Fig. 4. Proving hamiltonicity of NC(n, k) by induction.

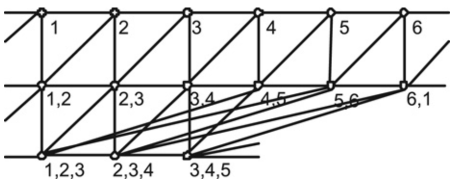

Fig. 5. The graph NC(6, 2). Only elements of one block are written.

Proof. For $n = 4$ the cycle is given by the following sequence of partitions:

$$\{1\}, \{1, 2\}, \{4\}, \{3\}, \{2\}, \{2, 3\}, \{1\}$$

(the corresponding graph is shown in Fig. 1).

Let us assume $n \geq 5$. Since no exchange of the element 1 is allowed, the cycle must avoid all the edges $(\{1, 2, \ldots, i\}, \{2, \ldots, i\})$, $(\{1, \ldots, i\}, \{2, \ldots, i + 1\})$, $(\{i, \ldots, n\}, \{i + 1, \ldots, n, 1\})$, $(\{i, \ldots, n, 1\}, \{i, \ldots, n\})$.

Therefore, the graph is split into two parts (refer to Fig. 6), a triangular part (the dashed edges in the figure) and a rectangular part (the solid edges in the figure) from which a triangle is cut off. The two parts are only connected by edges from the last level. In particular there are edges

$$(\{1, 2, \ldots, \lfloor n/2 \rfloor\}, \{\lfloor n/2 \rfloor + 2, \ldots, n\})$$

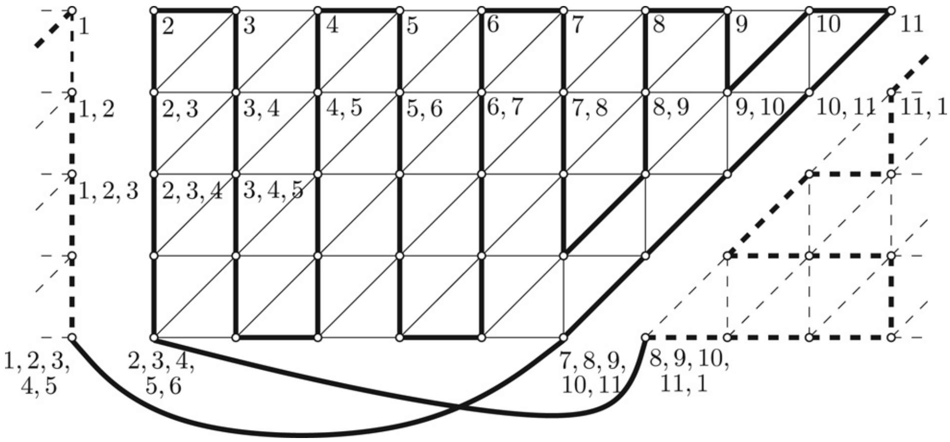

Fig. 6. A Hamilton cycle in the graph NC(11, 2).

and

$$(\{2, \dots, \lfloor n/2 \rfloor + 1\}, \{\lfloor n/2 \rfloor + 3, \dots, n, 1\})$$

which join the two parts. Edges which are not allowed in the cycle are omitted in Fig. 6. A cycle can be constructed in the following way: start at the partition $\{1\}$, walk down all levels to $\{1, 2, \dots, \lfloor n/2 \rfloor\}$, change to the other part of the graph by the edge

$$(\{1, 2, \dots, \lfloor n/2 \rfloor\}, \{\lfloor n/2 \rfloor + 2, \dots, n\})$$

and move slanting upwards to the partition {n} of the first level. From there it is possible to travel like in a "cogwheel" to the partition $\{2, \dots, \lfloor n/2 \rfloor + 1\}$, visiting all partitions of this part of the graph. According to the parity of n one can start by traversing the remaining nodes with either the sequence $\{n\}, \{n-1\}, \{n-2\}, \{n-1, n-2\}$ or with $\{n\}, \{n-1\}, \{n-1, n-2\}, \{n-2\}$ to ensure that the cogwheel way of visiting terminates in $\{2, \dots, \lfloor n/2 \rfloor + 1\}$. Then move back to the other part of the graph and move up the triangular part starting from $\{\lfloor n/2 \rfloor + 3, \dots, n, 1\}$ in zigzag. Finally, the edge connecting the partitions $\{n, 1\}$ and $\{1\}$ closes the cycle. Fig. 6 shows the whole construction of this Hamilton cycle for NC(11, 2). $\square$

2.1.2. *The graph* $\mathbf{NC}(\boldsymbol{n}, \boldsymbol{n} - 1)$

Every partition of the graph $\mathrm{NC}(n, n-1)$ consists of $n-2$ singletons and one doubleton. We denote the partition by just writing the doubleton.

Lemma 8. *For $n \geq 4$ there exists a Hamilton cycle for* $\mathrm{NC}(n, n-1)$ *in which no exchange of the element* 1 *is done and in which the partitions $\{1, n-1\}$ and $\{n-1, n\}$ are adjacent.*

Proof. A cycle fulfilling the conditions above is given by the following sequence of partitions

$$\underbrace{\{n, 2\}, \{n, 1\}, \{n, 3\}, \{n, 4\}, \dots, \{n, n-1\}}, \underbrace{\{n-1, 1\}, \{n-1, 2\}, \dots, \{n-1, n-2\}},$$
$$\underbrace{\{n-2, 1\}, \{n-2, 2\}, \dots, \{n-2, n-3\}}, \dots, \underbrace{\{3, 1\}, \{3, 2\}}, \underbrace{\{2, 1\}}$$

where in each underbraced group the first element in the subsets is the exchanged one. $\square$

This cycle for the graph NC(4, 3) can be seen in Fig. 7.

2.1.3. *Constructing Hamilton cycles by induction*

Given the sets $\mathrm{NC}(n, k)$ and $\mathrm{NC}(n, k+1)$ we describe how to use them in order to generate all partitions of $\mathrm{NC}(n+1, k+1)$. We divide the partitions of $\mathrm{NC}(n+1, k+1)$ into two classes. One class A contains all partitions in which 1 and $n+1$ belong to distinct blocks, the other class B is formed by partitions where 1 and $n+1$ are in the same block. The partitions of the first class will be derived from $\mathrm{NC}(n, k)$ and form a subgraph $\mathrm{NC}_A(n+1, k+1)$ of $\mathrm{NC}(n+1, k+1)$. The partitions of class B will be constructed from $\mathrm{NC}(n, k+1)$ and form the subgraph $\mathrm{NC}_B(n+1, k+1)$. By using Hamilton cycles in $\mathrm{NC}(n, k)$ and $\mathrm{NC}(n, k+1)$ we will create a Hamilton cycle in $\mathrm{NC}(n+1, k+1)$. In fact, for both subgraphs $\mathrm{NC}_A(n+1, k+1)$ and $\mathrm{NC}_B(n+1, k+1)$ we will construct a cycle induced by the Hamilton cycles in $\mathrm{NC}(n, k)$ and $\mathrm{NC}(n, k+1)$, respectively. Finally, we will connect these two cycles to get a Hamilton cycle for $\mathrm{NC}(n+1, k+1)$.

The graph $\mathbf{NC_A}(\boldsymbol{n} + 1, \boldsymbol{k} + 1)$

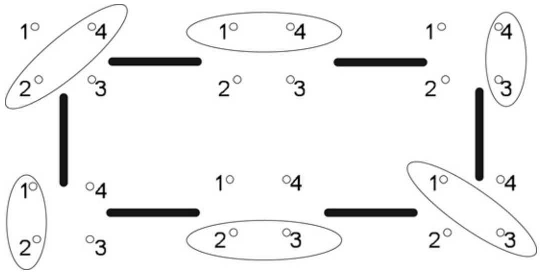

Fig. 7. A Hamilton cycle for the graph NC(4, 3).

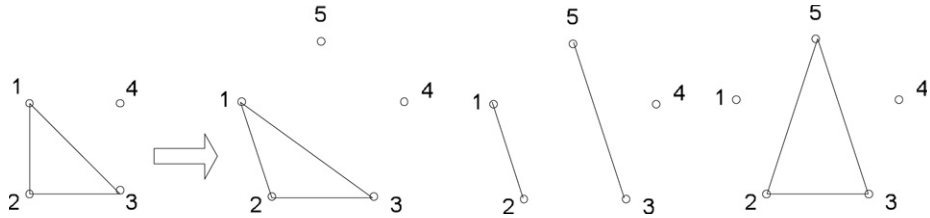

Fig. 8. Constructing the children of class A.

From every partition of NC(n, k) ("a parent") we construct several partitions of NC$(n+1, k+1)$ (its "children"). The children are generated in the following way: The first child is obtained by adding a new block $\{n+1\}$ to the partition. The following children are obtained by successively moving elements of the block which contains the element 1 to the new block of $\{n+1\}$. The element 1 always remains in the same block. Finally the last child contains the singleton $\{1\}$ and the new block contains $n+1$ and all elements which belonged to the block of 1.

Let us define the children formally. Given a partition $P = \{\{1, a_2, \ldots, a_j\}, \ldots\}$ of NC(n, k). The children of P are in NC$_A(n+1, k+1)$. The first child is defined as $C_1(P) = \{\{1, a_2, \ldots, a_j\}, \ldots, \{n+1\}\}$. In general, the child $C_l(P)$ is given by

$$\{\{1, a_2, \ldots a_{j-l+1}\}, \ldots, \{a_{j-l+2}, \ldots, a_j, n+1\}\}$$

for $l = 2, \ldots, j-1$. The last child is $C_j(P) = \{\{1\}, \ldots, \{a_2, \ldots, a_j, n+1\}\}$. In Fig. 8 the children in NC(5, 3) of the partition $P = \{\{1, 2, 3\}, \{4\}\} \in$ NC(4, 2) are shown.

Note that the children are well-defined, in the sense that every partition of NC$_A(n+1, k+1)$ has a parent and only one. It is obtained by joining the blocks of 1 and $n+1$ and deleting the element $n+1$. This parent is unique. Hence every partition of class A is generated exactly once.

In the following lemma, adjacency between two partitions is denoted with '$\sim$'.

Lemma 9. *Let $P, P_1, P_2 \in$ NC(n, k). The following properties hold:*

(a) *If $P_1 \sim P_2 \Rightarrow C_1(P_1) \sim C_1(P_2)$.*
(b) *If $P_1 \not\sim P_2$ through an exchange of the element 1 $\Rightarrow C_1(P_1) \not\sim C_1(P_2)$ through an exchange of 1.*
(c) *If $P_1 \sim P_2 \Rightarrow C_j(P_1) \sim C_j(P_2)$.*
(d) *If $P_1 \not\sim P_2$ through an exchange of the element 1 $\Rightarrow C_j(P_1) \not\sim C_j(P_2)$ through an exchange of 1.*
(e) *The children of P form a path having as extremes $C_1(P)$ and $C_j(P)$. In this path no exchange of the element 1 appears.*

This lemma can be proved easily because the first and the last child are essentially a copy of their parent. The path between the first and the last child is obtained from the construction of the children.

Lemma 10. *If there exists a Hamilton cycle in the graph* NC(n, k) *then there exists a Hamilton cycle in the graph* NC$_A(n+1, k+1)$*. Furthermore, if the cycle of* NC(n, k) *does not contain an exchange of the element 1, then also the cycle of* NC$_A(n+1, k+1)$ *does not.*

Proof. The first child of every partition of NC(n, k) can be seen as a copy of its parent because only the new block $\{n+1\}$ has been added. Thus all first children form a Hamilton cycle. Analogously there is a cycle for the last children. Every first child is connected with the last child through a path. Hence, NC$_A(n+1, k+1)$ has a cylindrical structure having the Hamilton cycles of the first and last children as bottom and top circles. There exists a circular tour through the cylinder moving like along a cogwheel. This tour is depicted in Fig. 9.

Note that there exist partitions in NC(n, k) which only have one child (when 1 is a singleton), i.e. their first and last child coincide. Thus one is able to choose the continuation of the tour to either the first or the last child of the next partition. Hence the cogwheel tour can always be closed, in spite of the parity of NC(n, k). If the cycle in NC(n, k) does not contain an exchange of the element 1 then also no edge of the cycle of NC$_A(n+1, k+1)$ corresponds to an exchange of 1 because of the previous lemma. $\square$

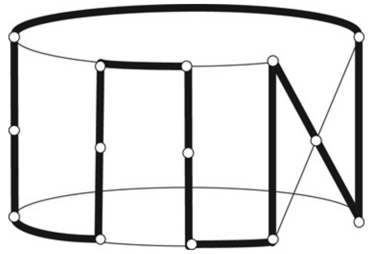

Fig. 9. Cogwheel tour in the graph $\mathrm{NC}_A(5, 3)$.

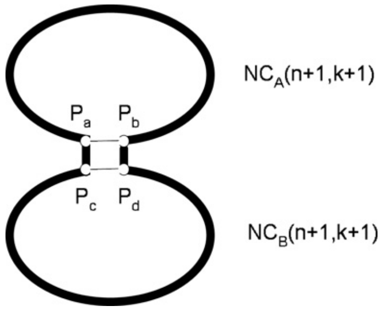

Fig. 10. Connecting two Hamilton cycles.

The graph $\mathbf{NC_B}(\boldsymbol{n}+1, \boldsymbol{k}+1)$

We generate all partitions of $\mathrm{NC}_B(n+1, k+1)$ from $\mathrm{NC}(n, k+1)$. For every partition of $\mathrm{NC}(n, k+1)$ we create a unique child which is defined in the following way: add the element $n+1$ to the block containing 1. Formally, if $P \in \mathrm{NC}(n, k+1) = \{\{1, a_2, \ldots, a_j\}, \ldots\}$ then the child $C(P) \in \mathrm{NC}_B(n+1, k+1)$ is defined as $C(P) = \{\{1, a_2, \ldots, a_j, n+1\}, \ldots\}$. For every partition in $\mathrm{NC}_B(n+1, k+1)$ there exists a unique parent which is obtained by deleting the element $n+1$ from its block.

Lemma 11. *If there exists a Hamilton cycle in* $\mathrm{NC}(n, k+1)$ *which does not contain an exchange of the element 1 then there exists a Hamilton cycle in* $\mathrm{NC}_B(n+1, k+1)$ *and this cycle does not contain an exchange of the element 1.*

Proof. Every child is like its parent with the addition of the new element $n+1$ to the block that contained the element 1. Since there is no exchange of the element 1, this element always remains in the same block and attaching $n+1$ to the element 1 does not affect any exchange in the former Hamilton cycle. Thus the graph $\mathrm{NC}_B(n+1, k+1)$ contains a Hamilton cycle which is essentially a copy of the cycle in $\mathrm{NC}(n, k+1)$. □

Connecting the cycles in $\mathbf{NC_A}(\boldsymbol{n}+1, \boldsymbol{k}+1)$ and $\mathbf{NC_B}(\boldsymbol{n}+1, \boldsymbol{k}+1)$

Next we show how to join the two cycles in $\mathrm{NC}_A(n+1, k+1)$ and $\mathrm{NC}_B(n+1, k+1)$. To do so we find partitions P_a and P_b which are adjacent in the cycle of $\mathrm{NC}_A(n+1, k+1)$ and partitions P_c and P_d which are adjacent in the cycle of $\mathrm{NC}_B(n+1, k+1)$. Then, if P_a is adjacent to P_c and P_b is adjacent to P_d, we can construct a cycle as depicted in Fig. 10.

We define the partitions P_a, P_b, P_c and P_d as follows:

$$P_a = \{\{1, n-1\}, B_2, \ldots, B_k, \{n, n+1\}\}$$
$$P_b = \{\{1\}, B_2, \ldots, B_k, \{n-1, n, n+1\}\}$$
$$P_c = \{\{1, n-1, n+1\}, B_2, \ldots, B_k, \{n\}\}$$
$$P_d = \{\{1, n+1\}, B_2, \ldots, B_k, \{n-1, n\}\}$$

where $B_2, \ldots, B_k$ are any blocks.

Lemma 12. (a) $P_a \sim P_b$ *in the Hamilton cycle of* $\mathrm{NC}_A(n+1, k+1)$ *that we have constructed in* Lemma 10 *for all* $n \geq 4$ *and* $2 \leq k \leq n-2$.
(b) $P_c \sim P_d$ *in the Hamilton cycle of* $\mathrm{NC}_B(n+1, k+1)$ *that we have constructed in* Lemma 11 *for all* $n \geq 4$ *and* $2 \leq k \leq n-2$.

Proof. P_a and P_b have the same parent, that is the partition $\{\{1, n-1, n\}, B_2, \ldots, B_k\}$. P_a is the second and P_b is the third child. By construction of the cycle $\mathrm{NC}_A(n+1, k+1)$ these partitions are adjacent. To see that $P_c \sim P_d$ in the cycle $\mathrm{NC}_B(n+1, k+1)$ let us examine the parents. The parent of P_c is $\{\{1, n-1\}, B_2, \ldots, B_k, \{n\}\}$ and will be denoted by P_e. The parent of P_d is $\{\{1\}, B_2, \ldots, B_k, \{n-1, n\}\}$ and is called P_f. If $P_e \sim P_f$ in the cycle $\mathrm{NC}(n, k+1)$ then $P_c \sim P_d$ in the cycle $\mathrm{NC}_B(n+1, k+1)$ because this cycle is just a copy of the former one (only the element $n+1$ is added). By Lemma 8 $P_e \sim P_f$ in the cycle of $\mathrm{NC}(n+1, k+1)$ for $n \geq 3$ and $k = n-1$. For $1 < k < n-1$ P_e and P_f have the same parent $\{\{1, n-1\}, B_2, \ldots, B_k\}$.

P_e and P_f are first and second child. By construction of the cycle these partitions are adjacent. Hence $P_e \sim P_f$ in the cycle of $\mathrm{NC}(n+1, k+1)$ for $n \geq 3$, $2 \leq k \leq n-1$. Therefore also $P_c \sim P_d$ in the cycle of $\mathrm{NC}_B(n+1, k+1)$ for $n \geq 4$ and $2 \leq k \leq n-2$. □

Therefore, as $P_a \sim P_c$ and $P_b \sim P_d$ we can always connect the cycles $\mathrm{NC}_A(n+1, k+1)$ and $\mathrm{NC}_B(n+1, k+1)$ and form a cycle in $\mathrm{NC}(n+1, k+1)$ as claimed. This concludes the proof of Theorem 6.

For the constrained graph $\mathrm{CNC}(n, k)$ we can answer negatively the particular case $k = 2$, and we conjecture that the graph is Hamiltonian for $k \geq 3$, but a proof remains elusive to us:

Proposition 13. *The graph* $\mathrm{CNC}(n, 2)$ *is not Hamiltonian for* $n \geq 4$.

Proof. Let $\sigma_i \in \mathrm{CNC}(n, 2)$ be the partition containing the sigleton $\{i\}$, and let $\delta_i \in \mathrm{NC}(n, 2)$ be the partition containing the doubleton $\{i, i+1\}$. Clearly σ_i is adjacent in $\mathrm{CNC}(n, 2)$ only to δ_{i-1} and δ_i (subscripts are taken modulo n), hence a Hamilton cycle should use necessarily these adjacencies. This means that the cycle

$$\sigma_1 \sim \delta_1 \sim \sigma_2 \sim \delta_2 \sim \cdots \sim \sigma_n \sim \delta_n \sim \sigma_1$$

must be in any Hamilton cycle, but this is only possible if $k = 3$, in which case $\mathrm{CNC}(n, 2)$ consists only of the σ_i and the δ_i. □

Conjecture 14. *For every* $k \geq 3$ *and every* $n > k$ *the graph* $\mathrm{CNC}(n, k)$ *contains a Hamilton cycle.*

3. Dissections of a convex polygon

Let P_n be a convex polygon with n vertices. A *dissection* of P_n is a subdivision of P_n into convex subpolygons by means of a set of k non-crossing diagonals, where $0 \leq k \leq n-3$. If $k = 0$ we have the empty dissection and if $k = n-3$ then we have a triangulation of P_n, that is, a dissection in which all the subpolygons are triangles. Dissections have been widely studied in combinatorics and geometry, see for instance [5] and [12]. We remark that $|\mathrm{D}(n, k)| = \frac{1}{k+1}\binom{n-3}{k}\binom{n+k-1}{k}$, a formula due to Kirkman.

Let $\mathrm{D}(n)$ and $\mathrm{D}(n, k)$ denote, respectively, the set of all dissections of P_n and those using exactly k diagonals. Define two dissections δ and δ' in $\mathrm{D}(n, k)$ adjacent if one can be obtained from the other by means of the following operation: remove a diagonal e from δ and add a new one inside the polygon formed by merging the two subpolygons adjacent to e. This defines a graph on the set of dissections with k diagonals that we denote also by $\mathrm{D}(n, k)$. The graph $\mathrm{D}(6, 2)$ is displayed in Fig. 11.

Our main result is as follows:

Theorem 15. *For every* $k \geq 1$ *and* $n \geq 5$ *the graph* $\mathrm{D}(n, k)$ *is Hamiltonian.*

Finally, we consider the possibility of removing a diagonal from a dissection without replacement (or reversely, adding a diagonal without producing any crossing). This defines a graph on $\mathrm{D}(n) = \cup \mathrm{D}(n, k)$ that again we denote by the same symbol $\mathrm{D}(n)$. By merging the Hamilton cycles obtained in the previous theorem we obtain our last result.

Theorem 16. *For every* $n \geq 4$ *the graph* $\mathrm{D}(n)$ *is Hamiltonian.*

3.1. *Proof of* Theorems 15 *and* 16

We prove Theorem 15 by induction from (n, k) to $(n+1, k+1)$.

Let P_n be a convex polygon with n vertices $\{p_1, p_2, \ldots, p_n\}$ in counterclockwise order and let $D(n, k)$ be the set (or the graph, respectively) of dissections having k diagonals on top of P_n. Assume, we are given the set of dissections $D(n, k)$. We will obtain all dissections of $D(n+1, k+1)$ as described in the following. To every dissection δ of $D(n, k)$ we assign several dissections of $D(n+1, k+1)$. δ will be called *parent* and its assigned dissections will be called *children*. Every dissection $\delta^\star$ of $D(n+1, k+1)$ will have a unique parent and every dissection of δ will have children. We define this assignment by showing how the parent of $\delta^\star$ is obtained:

Starting at point p_n on the convex polygon we walk along the boundary of the polygon in counterclockwise order until we meet the first diagonal. We delete this diagonal. Finally, the edge $p_n p_{n+1}$ is contracted to a point. The resulting dissection is in $D(n, k)$ because exactly one point and one diagonal have been removed. Fig. 12 shows the assignment of a parent. Once we know how to find a parent of a dissection we can derive rules for generating the children of a dissection δ of $D(n, k)$. δ has r diagonals incident to point p_n. We create children of different types t, $0 \leq t \leq r$. First, the new point p_{n+1} is inserted into the dissection. This is done by splitting the edge $p_1 p_n$. Then we move t diagonals incident to p_n to the new point p_{n+1}. That is, a diagonal $p_a p_n$ is replaced by the diagonal $p_a p_{n+1}$. Finally, we insert a new diagonal into the subpolygon Q which contains p_n and p_{n+1}. This new diagonal must be the first diagonal which is met when walking in counterclockwise order along the border of the polygon starting from p_n. Hence, for the children of type 0 the new diagonal either is incident to p_{n+1} or the new diagonal $p_i p_j$ with $i < j$ fulfills $i \in \{1, 2, \ldots, k\}$ such that k is the smallest index of a point of Q which is incident to a diagonal in $D(n, k)$. For children of type $s > 0$ the new diagonal is always incident to p_{n+1}. It is easy to see that this

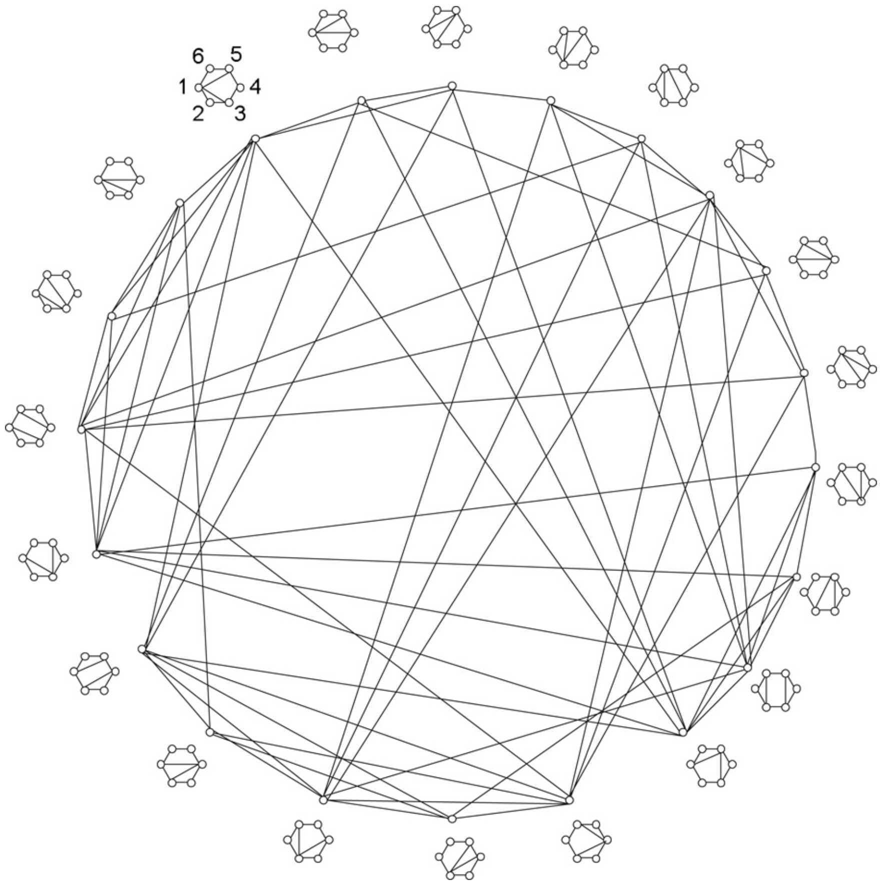

Fig. 11. The graph $D(6, 2)$.

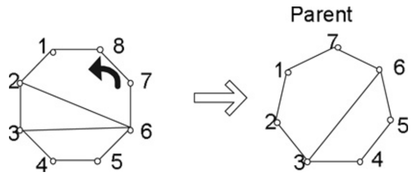

Fig. 12. Assigning the parent to a dissection in D(8,2).

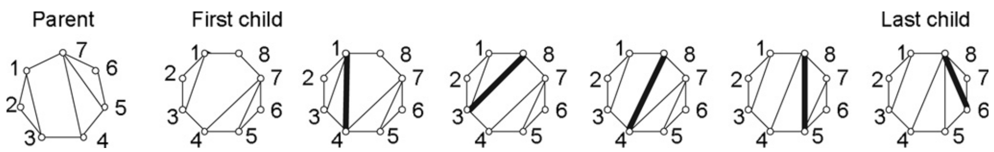

Fig. 13. A dissection of D(7,3) and its children.

generation rule for the children corresponds to the previous defined way of finding the parent. Fig. 13 shows the children of a dissection. The new inserted diagonal is drawn in bold. The first four children are of type 0, one child is of type 1 and the last child is of type 2. We will show that the children of a dissection δ of $D(n, k)$ form a Hamilton path whose endpoints are essentially a copy of δ. We call the child of type 0 which has the diagonal p_1p_n *first child* of δ. It is denoted with δ_f. We call the child of the last type r with new diagonal $p_{n-1}p_{n+1}$ *last child* of δ. It is denoted with δ_l. The *first child of type* s, $0 < s \leq r$, is the child of type s for which the new inserted diagonal is lexicographically minimal. Analogously, for the *last child of type* s, $0 \leq s < r$, the new inserted diagonal is lexicographically maximal.

In the following adjacency between two dissections will be denoted by $\sim$. The proof of the following lemma is straightforward and omitted.

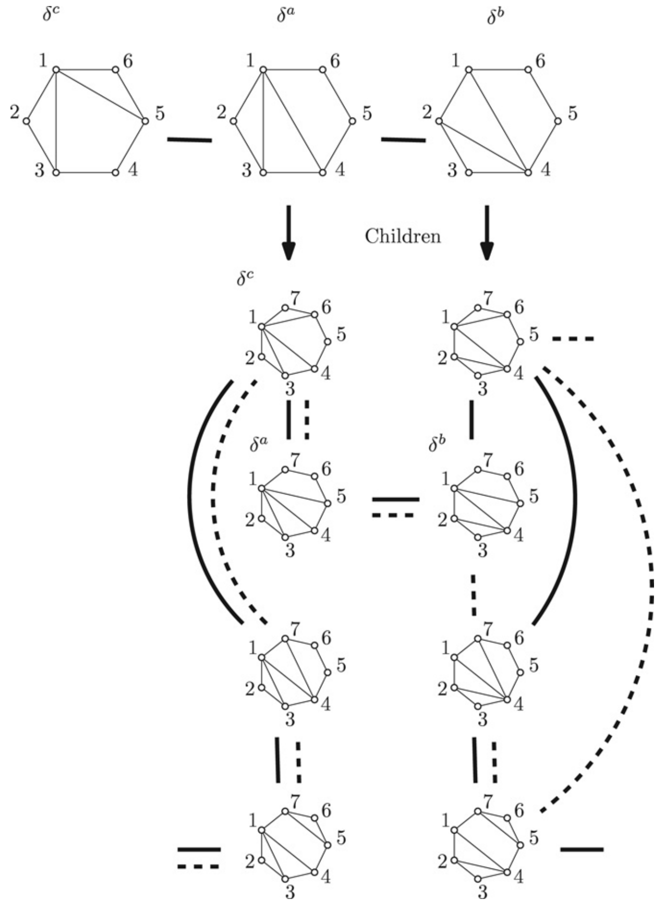

Fig. 14. Two Hamilton paths for the children of δ^a and δ^b.

Lemma 17. *Let δ be a dissection of $D(n, k)$ with $0 \le k \le n-3$. Let r denote the number of diagonals incident to p_n. The children of δ have the following properties:*

(a) *The children of type s, $0 \le s \le r$, form a complete graph.*
(b) *The first child δ_f is a copy of δ. That is, if δ is adjacent to a dissection δ', then $\delta_f \sim \delta'_f$.*
(c) *The last child δ_l is a copy of δ. That is, if δ is adjacent to a dissection δ', then $\delta_l \sim \delta'_l$.*
(d) *The last child of type s is adjacent to every child of type $s+1$ for $0 \le s \le r-1$.*
(e) *The children of δ form a Hamilton path whose endpoints are the first child δ_f and the last child δ_l.*

We identify three special dissections δ^a, δ^b and δ^c, which will be needed for the construction of a Hamilton cycle.

$\delta^a \in D(n, k)$ contains k diagonals $p_1p_i : i \in \{3, 4, \ldots, k+2\}$

$\delta^b \in D(n, k) = \delta^a \setminus \{p_1p_3\} \cup \{p_2p_4\}$

$\delta^c \in D(n, k)$ contains $k-1$ diagonals $p_1p_i : i \in \{3, 4, \ldots, k+1\}$ and the diagonal p_1p_{n-1}.

δ^c, δ^a and δ^b of $D(6, 2)$ and the children of δ^a and δ^b are depicted in Fig. 14.

The following properties hold:

Observation 18. *Let $\delta^a, \delta^b \in D(n, k)$.*

All children of δ^a are of type 0. Thus, they form a complete graph.

All children of δ^b are of type 0. Thus, they form a complete graph.

Every child of δ^a is adjacent to exactly one child of δ^b.

$\delta^a \in D(n+1, k+1)$ is a child of $\delta^a \in D(n, k)$.

$\delta^b \in D(n+1, k+1)$ is a child of $\delta^b \in D(n, k)$.

$\delta^c \in D(n+1, k+1)$ is the first child of $\delta^a \in D(n, k)$.

The statement of Theorem 15 follows from Lemmas 19 and 21.

Lemma 19. *The graph $D(n, k)$ is Hamiltonian for $n \ge 5$ and $1 \le k \le n-4$. More precisely, there is a Hamilton cycle in which δ^c, δ^a and δ^b are consecutive elements.*

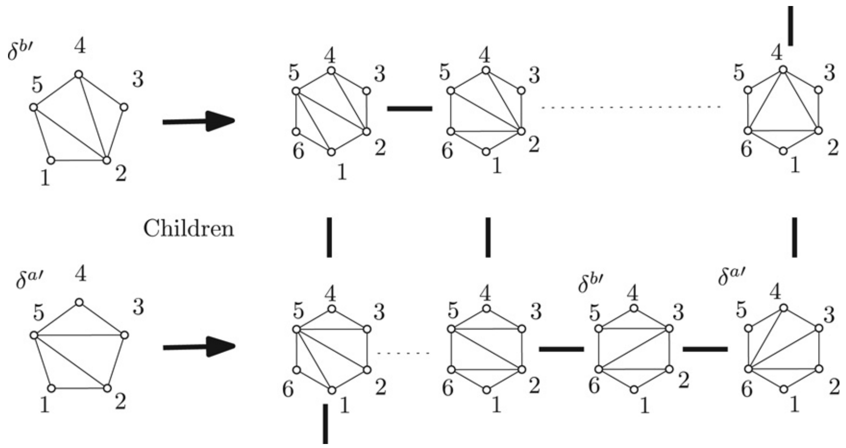

Fig. 15. A Hamilton path formed by the children of $\delta^{a\prime}$ and $\delta^{b\prime}$.

Proof. We proceed by induction from (n, k) to $(n+1, k+1)$. $D(n, 1)$ is a complete graph. The demanded Hamilton cycle clearly exists. Assume we are given a Hamilton cycle in $D(n, k)$ in which δ^c, δ^a and δ^b are consecutive elements. Due to Lemma 17 the children of every dissection of $D(n, k)$ form a Hamilton path. We can connect the paths formed by the children of adjacent dissections of the given Hamilton cycle. For that, we use the adjacencies of the first children and the last children, respectively (see Lemma 17). A path of children which starts at the first child is followed by a reversed path (i.e. starting at the last child). This construction gives a Hamilton path in $D(n+1, k+1)$. In order to be able to always obtain a Hamilton cycle, we connect the children of δ^a and δ^b in an adequate manner according to the parity of $D(n, k)$. The two possible paths are shown in Fig. 14 for the case $n = 6$ and $k = 2$. The bold solid lines show the Hamilton path which is used when $|D(n, k)|$ is even. The endpoints of this path are the last child of δ^a and the last child of δ^b. If $|D(n, k)|$ is odd the Hamilton path given by the bold dashed edges is used. The endpoints of this Hamilton path are the last child of δ^a and the first child of δ^b. Both Hamilton paths always exist because of Observation 18. In the obtained Hamilton cycle the corresponding δ^c, δ^a and δ^b of $D(n+1, k+1)$ are consecutive elements. $\square$

This proof does not work for $k = n-3$, when we deal with triangulations. In this case each of δ^a and δ^b only has two children. These do not form a path with endpoints δ^a_l and δ^b_f. Furthermore, δ^a and δ^c coincide. Therefore, we define other dissections $\delta^{a\prime}$ and $\delta^{b\prime}$ for triangulations.

$\delta^{a\prime} \in D(n, k)$ has $n-3$ diagonals $p_i p_n : i \in \{2, 3, \ldots, n-2\}$.

$\delta^{b\prime} \in D(n, k)$ has $n-4$ diagonals $p_i p_n : i \in \{2, 3, \ldots, n-3\}$ and the diagonal $p_{n-3}p_{n-1}$.

These two dissections and their children are shown in Fig. 15 for $n = 5$.$\delta^{a\prime}$ and $\delta^{b\prime}$ have the following properties. Again, the proof is straightforward and omitted.

Lemma 20. *Let $\delta^{a\prime}$ and $\delta^{b\prime}$ be in $D(n, k)$.*

(a) *$\delta^{a\prime}$ and $\delta^{b\prime}$ are adjacent.*
(b) *$\delta^{a\prime}$ has $n-1$ children: Two of type 0 and one for every diagonal incident to n.*
(c) *$\delta^{a\prime}$ in $D(n+1, k+1)$ is the last child of $\delta^{a\prime}$. It is of type $n-3$.*
(d) *$\delta^{b\prime}$ in $D(n+1, k+1)$ is the unique child of type $n-4$ of $\delta^{a\prime}$.*
(e) *$\delta^{b\prime}$ has $n-2$ children: Two of type 0 and one for every diagonal incident to n.*
(f) *Let $\delta^{a\prime}_i$ denote the unique child of type i of $\delta^{a\prime}$ and let $\delta^{b\prime}_i$ denote the unique child of type i of $\delta^{b\prime}$. Then, $\delta^{a\prime}_i \sim \delta^{b\prime}_i$, for $1 \le i \le n-5$.*
(g) *The first children of $\delta^{a\prime}$ and $\delta^{b\prime}$ are adjacent. Also the other children of type 0 are adjacent.*
(h) *The last children of $\delta^{a\prime}$ and $\delta^{b\prime}$ are adjacent.*

We now prove the missing case $k = n-3$ of Theorem 15.

Lemma 21. *$D(n, n-3)$ is Hamiltonian for $n \ge 5$. Moreover, there is a Hamilton cycle in which $\delta^{a\prime}$ and $\delta^{b\prime}$ are adjacent.*

Proof. The construction of the Hamilton cycle is the same as for the previous proof of Lemma 19. We just have to care about how to connect the children of $\delta^{a\prime}$ and $\delta^{b\prime}$.

If $|D(n, k)|$ is even, we connect the paths formed by the children of $\delta^{a\prime}$ and $\delta^{b\prime}$ via the first children $\delta^{a\prime}_f$ and $\delta^{b\prime}_f$ or via the last children $\delta^{a\prime}_l$ and $\delta^{b\prime}_l$. $\delta^{a\prime}$ and $\delta^{b\prime}$ of $D(n+1, k+1)$ are adjacent in the obtained Hamilton cycle. If $|D(n, k)|$ is odd, the desired path is given by the sequence of dissections

$$\delta^{a\prime}_f, \delta^{b\prime}_f, \delta^{b\prime}_{0,1}, \delta^{a\prime}_{0,1}, \delta^{a\prime}_1, \delta^{b\prime}_1, \delta^{b\prime}_2, \delta^{a\prime}_2, \delta^{a\prime}_3, \delta^{b\prime}_3, \ldots, \delta^{b\prime}_{n-5}, \delta^{a\prime}_{n-5}, \delta^{a\prime}_{n-4}, \delta^{a\prime}_l, \delta^{b\prime}_l.$$

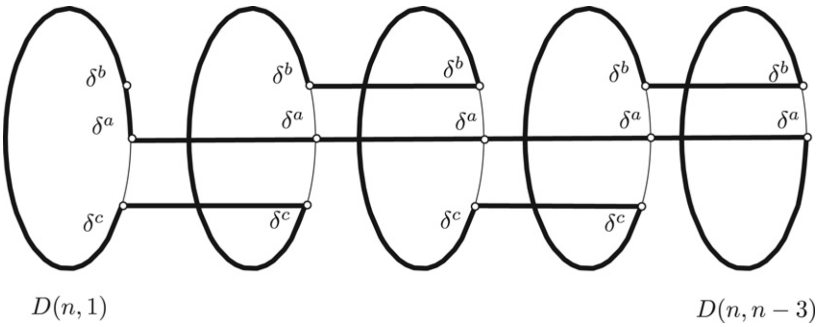

Fig. 16. The construction of a Hamilton cycle in the graph $D(n)$.

This path can be seen as a 'zigzag' walk, moving from the first child of $\delta^{a\prime}$ to the last child of $\delta^{b\prime}$ and alternating between the children of $\delta^{a\prime}$ and $\delta^{b\prime}$ (see Fig. 15). Note that this construction only gives a Hamilton path if n is odd. It is well known that if $|D(n, n-3)|$ is odd, then also n is odd. Therefore, we can always construct a Hamilton cycle, in which $\delta^{a\prime}$ and $\delta^{b\prime}$ are adjacent. □

Corollary 22. *The graph $D(n, n-3)$ contains a Hamilton cycle in which δ^a and δ^b are adjacent.*

Proof. We have constructed a Hamilton cycle for triangulations in which $\delta^{a\prime}$ and $\delta^{b\prime}$ are adjacent. If we relabel the points of the convex polygon we get another Hamilton cycle. In particular, we relabel the convex polygon such that p_n will be p_1 and then we label the points in clockwise order. We observe that $\delta^{a\prime}$ is transformed into δ^a and $\delta^{b\prime}$ is transformed into δ^b by this operation. Hence, in the obtained Hamilton cycle $\delta^a \sim \delta^b$. □

The statement of Theorem 16 follows from Lemma 19, Corollary 22 and Observation 23. In the graph $D(n)$ also edge-inserting and edge-removing flips are allowed.

Observation 23. *$\delta^a \in D(n, k)$ is adjacent to $\delta^a \in D(n, k+1)$ for $1 \le k \le n-4$.*
$\delta^b \in D(n, k)$ is adjacent to $\delta^b \in D(n, k+1)$ for $1 \le k \le n-4$.
$\delta^c \in D(n, k)$ is adjacent to $\delta^c \in D(n, k+1)$ for $1 \le k \le n-5$.

Proof of Theorem 16. We use the obtained Hamilton cycles of $D(n, k)$ for $1 \le k \le n-3$ to construct a Hamilton cycle in $D(n)$. These cycles are connected by δ^a, δ^b and δ^c, as indicated by Observation 23. Edge-inserting or edge-removing flips are only applied to connect two of these dissections. Fig. 16 shows the construction of this cycle. Finally, the dissection $D(n, 0)$ can be placed between any two consecutive dissections of $D(n, 1)$ in this Hamilton cycle.

Acknowledgments

We thank Regina Estkowski for useful discussions on early versions of this paper. First and second authors were partially supported by projects MCYT BFM2003-0368, MEC MTM2006-01267 and DURSI 2005SGR00692. The third author was partially supported by project MTM2005-08618-C02-01.

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